2023

PHYSICS — HONOURS

Paper : DSE-A1.1 and DSE-A-1.2

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Paper : DSE-A1.1

(Advanced Mathematical Methods)

Full Marks : 65

Answer ***question nos.*** **1** and **2**, and ***any four*** questions from the rest (***Q.*** **3** to ***Q.*** **8**).

1. Answer ***any five*** questions : 2×5

(a) A binary operation is defined as : $a * b = a^2 + b^2$ $(a, b \in \mathbb{R})$. Check commutativity and associativity of the operation.

(b) Find out if W is a subspace of vector space R^n, $(n \geq 3)$, where $W = \{\alpha : \alpha \in R^n \text{ and } a_1 + 3a_2 = a_3\}$.

(c) If P_1 and P_2 are two projection operators, then under what condition is $P_1 + P_2$ also a projection operator?

(d) Show that $\vec{\nabla} \times \vec{\nabla}\phi = 0$ using index notation, where ϕ is a scalar field.

(e) Show that a 2nd rank covariant symmetric tensor remains symmetric under a general coordinate transformation.

(f) For what real values of k does the set S form a basis of $\mathbb{R}^3$:

$$S = \{(k, 0, 1), (1, k+1, 1), (1, 1, 1)\}.$$

(g) Define equivalent representations in group theory.

2. Answer ***any three*** questions :

(a) Find out if the following mappings are linear transformation (homomorphism) or not.

(i) $F : R^2 \to R^2$ is defined by $F(x, y) = (x + y, x)$.

(ii) $F : R^3 \to R^2$ is defined by $F(x, y, z) = (|x|, y + z)$. 2+3

(b) Find $\cos\theta$, where θ is the angle between :

(i) $u = (1, 3, -5, 4)$ and $v = (2, -3, 4, 1)$ in R^4.

(ii) $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, where $\langle A | B \rangle = tr\left(B^T A\right)$. 2+3

(c) Write down the general expression for moment of inertia I_{ij}. Show that I_{ij} transforms as a 2nd rank tensor under rotation in R^3. 2+3

(d) (i) Show that all $(n \times n)$ unitary matrices form a group under multiplication.

(ii) Show that all such matrices with determinant 1 form a subgroup. 3+2

(e) Suppose $\{e, a, b\}$ forms a group under multiplication, where e is the identity element. Construct the group multiplication table. Find the inverse element of a and b. 4+1

3. (a) Show that a set of orthogonal vectors is linearly independent.

(b) Find a unit vector orthogonal to the vectors $\alpha_1 = (0, 2, 1)$ and $\alpha_2 = (3, 1, 2)$.

(c) (i) A linear transformation T in $\mathbb{R}^2$ is defined as $T|e_1\rangle = |e_2\rangle$ and $T|e_2\rangle = -|e_1\rangle$. Write down the matrix representation of T in $\{|e_1\rangle, |e_2\rangle\}$ basis.

(ii) Let $|\alpha_1\rangle = |e_1\rangle + |e_2\rangle$ and $|\alpha_2\rangle = -|e_1\rangle$ be another basis. Write down the matrix representation of T relative to $\{|\alpha_1\rangle, |\alpha_2\rangle\}$ basis. 3+2+(2+3)

4. (a) Let $V = P_2(t)$ with inner product $\langle f | g \rangle = \int_0^1 f(t)g(t)dt$.

(i) Find $\langle f | g \rangle$, where $f(t) = t + 2$ and $g(t) = t^2 - 3t + 4$.

(ii) Find the matrix A of the inner product with respect to the basis $\{1, t, t^2\}$ of V, where $A_{ij} = \langle e_i | e_j \rangle$, e_i's are elements of V.

(b) Let U be the subspace of R^4 spanned by $v_1 = (1, 1, 1, 1)$, $v_2 = (1, -1, 2, 2)$ and $v_3 = (1, 2, -3, -4)$.

(i) Apply Gram-Schmidt algorithm to find an orthogonal and orthonormal basis for U.

(ii) Find the projection of $v = (1, 2, -3, 4)$ onto U. (2+2)+(4+2)

5. (a) Derive the metric tensor for 3D spherical polar coordinate.

(b) Write the transformation rule for mixed tensor of rank two.

(c) Using the expressions of Lorentz transformation, obtain the Lorentz transformation matrix. At which limit, the Lorentz transformation becomes similar to Galilean transformation? 3+2+(4+1)

6. Maxwell's equations in covariant form is represented as $\partial_\mu F^{\mu\nu} = j^\nu$.

(a) Show that $\partial_\nu j^\nu = 0$ using antisymmetry of $F^{\mu\nu}$.

(b) Given $F^{0i} = E^i$, $(i = 1, 2, 3)$ and $F^{ij} = \epsilon^{ijk} B_k$, construct the $F^{\mu\nu}$ matrix in E^i's and B^j's. From inhomogeneous Maxwell's equations $\partial_\mu F^{\mu\nu} = j^\nu$, show that its zeroth component yields $\nabla . \vec{E} = \frac{\rho}{\epsilon_0}$.

(c) Construct $F_{\mu\nu}$ from $F^{\mu\nu}$. (use the signature $\{1, -1, -1, -1\}$) and calculate $F_{\mu\nu} F^{\mu\nu}$.

2+(2+2)+(2+2)